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POINT RESONANCE IN A SYSTEM OF TWO OSCILLATORS*

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The problem of deriving asymptotic solutions for a system of equations of two nonlinear oscillators whose frequencies are linked by resonance relation only at some instant of time is analyzed. Unlike the combined asymptotics obtained in /1/ by the method of multiple scales, a global asymptotics, defined by a single formula in both the resonance and nonresonance zones, is obtained here.

The problem of constructing asymptotic solutions for the system

$$x^{\cdot\cdot} + a^2(\varepsilon t)\, x = \varepsilon y^2,\ x(0, \varepsilon) = x^0,\ x^{\cdot}(0, \varepsilon) = v^0 \tag{0.1}$$
$$y^{\cdot\cdot} + b^2(\varepsilon t)\, y = 2\varepsilon xy,\ y(0, \varepsilon) = y^0,\ y^{\cdot}(0, \varepsilon) = z^0$$

where $\varepsilon > 0$ is a small parameter, and frequencies a and b of oscillators are either constant or weakly dependent on time was considered in many publications /1-3/. For instance, system (0.1) with constant a and b defines the motion of stars in the Galaxy (see, e.g., /2/). Application of the theory of asymptotic solutions to systems of the type (0.1) becomes rather complicated when a and b are variable, particularly in the case of the so-called point resonance, which is defined as follows. Let the variable t in system (0.1) vary over an asymptotically large time interval $[0, l/\varepsilon]$, where $l > 0$ is a constant. We introduce the slow time $\tau = \varepsilon t$, and say that a point resonance occurs in system (0.1), when at some $\tau = \tau_0 \in [0, l]$ the oscillator frequencies a and b (0.1) are related by the formula

$$2b(\tau) - a(\tau) = 0 \tag{0.2}$$

It is assumed that when $\tau \in [0, l]$ and $\tau \neq \tau_0$, relation (0.2) cannot be satisfied. In the case of point resonance the combined frequency $a(\tau) - 2b(\tau)$ can be represented in the form

$$a(\tau) - 2b(\tau) \equiv (\tau - \tau_0)^r\, \psi(\tau),\ \psi(\tau) \neq 0,\ \forall \tau \in [0, l] \tag{0.3}$$

where r is a positive integer.

A procedure for constructing asymptotic solutions by the method of multiple scales /2/ was derived in /1/ for system (0.1) in the case of $r = 1$. The proposed here procedure involves the subdivision of segment $[0, l]$ into three zones: preresonance, resonance, and postresonance zones. In each of these zones we construct its own expansion of system (0.1). The merging of these expansions enables us to obtain the composite asymptotics that is uniformly applicable over the whole segment $[0, l]$.

The development of an algorithm which would avoid the subdivision of the segment into zones and, thus, simplify the process of constructing asymptotics is of interest. Such an algorithm is proposed here. The theory of regularized asymptotic solutions developed in this connection is an extension of the regularization method /4/ to the case of point resonance.

1. Regularization of system (0.1) and solvability of iterational problems.

By the substitution of variables $\tau = \varepsilon t$, $\varepsilon y' = z$, $\varepsilon x' = v$, where the prime denotes a derivative with respect to τ, we pass from problem (0.1) to the problem

$$\varepsilon dw/d\tau = A(\tau)\, w + F(w),\ w(0, \varepsilon) = w^\circ,\ w = \{y, z, x, v\} \tag{1.1}$$
$$F(w) = \{0, 2xy, 0, y^2\},\quad A(\tau) = \begin{Vmatrix} A_1 & 0 \\ 0 & A_2 \end{Vmatrix},\quad A_1 = \begin{Vmatrix} 0 & 1 \\ -b^2 & 0 \end{Vmatrix}$$
$$A_2 = \begin{Vmatrix} 0 & 1 \\ -a^2 & 0 \end{Vmatrix},\quad w^\circ = \{y^\circ, z^\circ, x^\circ, v^\circ\}$$

In what follows we shall need the eigenvalues $\lambda_j(\tau)$ and eigenvectors $c_j(\tau)$ of matrix $A(\tau)$. It will be readily seen that

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$$\begin{aligned}
&\lambda_1(\tau) = -b(\tau)\, i,\ \lambda_2(\tau) = +b(\tau)\, i \\
&\lambda_3(\tau) = -a(\tau)\, i,\ \lambda_4(\tau) = +a(\tau)\, i \\
&c_1(\tau) = \{-ib,\ -b^2,\ 0,\ 0\},\ c_2(\tau) = \{+ib,\ -b^2,\ 0,\ 0\} \\
&c_3(\tau) = \{0,\ 0,\ -ia,\ -a^2\},\ c_4(\tau) = \{0,\ 0,\ +ia,\ -a^2\}
\end{aligned} \tag{1.2}$$

(here and subsequently braces denote column vectors and parentheses row vectors). We shall further require eigenvectors $d_j(\tau)$ of the adjoint matrix $A^*(\tau)$

$$\begin{aligned}
&d_1(\tau) = \{+ib,\ 1,\ 0,\ 0\},\ d_2(\tau) = \{-ib,\ 1,\ 0,\ 0\} \\
&d_3(\tau) = \{0,\ 0,\ +ia,\ 1\},\ d_4(\tau) = \{0,\ 0,\ -ia,\ 1\}
\end{aligned} \tag{1.3}$$

To derive the asymptotic solution of system (1.1) we apply the nonlinear variant of the regularization method /5/, in conformity with which we regularize problem (1.1) using vector $u = \{u_1, u_2, u_3, u_4\}$ of the regularizing functions that satisfy the system

$$\begin{aligned}
&\varepsilon \frac{du}{d\tau} = \Lambda(\tau) u + \sum_{k=1}^{m+1} \varepsilon^k g_k(\tau, u), \quad u(0, \varepsilon) = \mathbf{1} \\
&\Lambda(\tau) = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\},\ \mathbf{1} = \{1, 1, 1, 1\}
\end{aligned} \tag{1.4}$$

where $g_k(\tau, u)$ are some functions nonlinear with respect to u, which will be defined below, and $k = 1, 2, \ldots, m + 1$.

We shall call (1.4) the regularizing system of order $m + 1$. It is selected so as to enable its regularization by the method of /4/.

We introduce now the widening $w_*(\tau, u, \varepsilon)$ of solution $w(\tau, \varepsilon)$ of problem (1.1) which satisfies the system of equations

$$\varepsilon \frac{\partial w_*}{\partial \tau} + \frac{\partial w_*}{\partial u}\left[\Lambda(\tau) u + \sum_{k=1}^{m+1} \varepsilon^k g_k(\tau, u)\right] = A(\tau) w_* + \varepsilon F(w_*), \quad w_*(0, \mathbf{1}, \varepsilon) = w^0 \tag{1.5}$$

The contraction $w_*(\tau, u(\tau, \varepsilon), \varepsilon)$ of function $w_*(\tau, u, \varepsilon)$ on solution $u = u(\tau, \varepsilon)$ of the regularizing system (1.4) is the same as the exact solution $w(\tau, \varepsilon)$ of the input problem (1.1). However, unlike the latter, problem (1.5) is regular with respect to ε, as $\varepsilon \to +0$), hence its solution can be defined in the form of series

$$w_*(\tau, u, \varepsilon) = \sum_{k=0}^{\infty} w_k(\tau, u)\, \varepsilon^k \tag{1.6}$$

Substituting (1.6) into (1.5) and equating coefficients at like powers of ε, we obtain for $w_k(\tau, u)$ the following problems:

$$\begin{aligned}
&Lw_0 \equiv \frac{\partial w_0}{\partial u} \Lambda(\tau) u - A(\tau) w_0 = 0, \quad w_0(0, \mathbf{1}) = w^0 \\
&Lw_1 = -\frac{\partial w_0}{\partial \tau} - \frac{\partial w_0}{\partial u} g_1(\tau, u) + F(w_0), \quad w_1(0, \mathbf{1}) = 0 \\
&\ldots
\end{aligned}$$

Let us define the space U of solutions of iterational problems (ε^k), $k \geqslant 0$.

Definition 1. The monomial $w^{(m)}(\tau) \equiv w^{(m_1, \ldots, m_4)}(\tau)\, u_1^{m_1} \ldots u_4^{m_4}$ with vector coefficient $w^{(m)}(\tau) = \{w_1^{(m)}, \ldots, w_4^{(m)}\}$ is the λ_s-resonance monomial $(s = 1, \ldots, 4)$, if its vector index $m = (m_1, \ldots, m_4)$ satisfies, for at least one $\tau \in [0, l]$, the relation

$$(m, \lambda(\tau)) \equiv \sum_{j=1}^{4} m_j \lambda_j(\tau) = \lambda_s(\tau), \quad |m| = \sum_{j=1}^{4} m_j \geqslant 2 \tag{1.7}$$

where $(m_1, \ldots, m_4$ are nonnegative integers.

Definition 2. We say that the λ_s-resonance monomial $w^{(m)}(\tau)\, u^m$ is orthogonalized, if

$$(w^{(m)}(\tau), d_s(\tau)) \equiv 0, \quad \forall \tau \in [0, l] \tag{1.8}$$

where $d_s(\tau)$ is the eigenvector of matrix $A^*(\tau)$ that corresponds to the eigenvalue $\bar{\lambda}_s(\tau)$ and (,) is the usual scalar product in the complex space $\mathbf{C}_\tau^4$ (the bar over λ_s denotes complex conjugation and the subscript τ in $\mathbf{C}_\tau^4$ indicates that the respective vectors are for fixed $\tau \in [0, l]$).

Definition 3. We say that function $w(\tau, u) = \{w_1, \ldots, w_4\} \in U$, if $w(\tau, u)$ is some polynomial in u with coefficients from class $C^\infty[0, l]$ in which all monomials are orthogonalized.

Solution of the iterational problem (ε^0) in space U is of the form of a linear function in u, which can be written as follows:

$$w_0(\tau, u) = \sum_{k=1}^{4} \alpha_k(\tau) c_k(\tau) u_k \tag{1.9}$$

where $c_k(\tau)$ are eigenvectors (1.2) of matrix $A(\tau)$ and $\alpha_k(\tau)$ are arbitrary scalar functions of class $C^\infty[0, l]$ that satisfy only the condition $w_0(0, 1) = w^0$.

For $k > 0$ the iterational problems (ε^k) are of the form

$$Lw(\tau, u) = -(\partial w_0/\partial u)\, g(\tau, u) + h(\tau, u),\quad w(0, 1) = 0 \tag{1.10}$$

where $h(\tau, u)$ is some polynomial in u with coefficients of class $C^\infty[0, l]$. Since generally the polynomial $h(\tau, u)$ does not belong to space U, hence the question of solvability of system (1.10) in that space has no meaning, as long as the right-hand part of system (1.10) is not imbedded in U. We use function $g(\tau, u)$ for effecting the imbedding into space U. We select that function in the form of sum of resonance monomials ($g_s^{(m^j)}$ with (τ) as the s-component of vector $g^{(m^j)}(\tau)$)

$$g(\tau, u) = \sum_{j=1}^{4} \sum_{|m^j| \geqslant 2,\ (m^j, \lambda(\tau)) = \lambda_j(\tau)} g^{(m^j)}(\tau)\, u^{m^j} \tag{1.11}$$

$$g^{(m^j)}(\tau) \in C^\infty[0, l],\quad g_s^{(m^j)}(\tau) \equiv 0,\ s \neq j,\ j, s = 1, \ldots, 4$$

Theorem 1. Let the frequencies $a(\tau)$ and $b(\tau) \in C^\infty[0, l]$ satisfy condition (0.2) of point resonance and $\alpha_i(\tau) \neq 0$, $\forall \tau \in [0, l]$, $i = 1, \ldots, 4$. Let also $h(\tau, u)$ be a polynomial in u with coefficients of class $C^\infty[0, l]$. There exists then a unique function $g(\tau, u)$ of the form (1.11) in which summation is carried out over all different resonance monomials of the polynomial $h(\tau, u)$ such that

$$-(\partial w_0/\partial u) g(\tau, u) + h(\tau, u) \in U \tag{1.12}$$

where $w_0(\tau, u)$ is some fixed solution (1.9) of problem (ε^0).

Proof. Let the sum of all resonance monomials of the polynomial $h(\tau, u)$ be of the form

$$\sum_{j=1}^{4} \sum_{|m^j| \geqslant 2,\ (m^j, \lambda(\tau)) = \lambda_j(\tau)} h^{(m^j)}(\tau)\, u^{m^j}$$

We denote by $C(\tau)$ the matrix of eigenvectors $c_j(\tau)$, $j = 1, \ldots, 4$. Conditions (1.12) are obviously equivalent to conditions

$$(-C(\tau)\, G(\tau)\, g^{(m^j)}(\tau) + h^{(m^j)}(\tau),\ d_j(\tau)) \equiv 0,\ j = 1, \ldots, 4 \tag{1.13}$$
$$G(\tau) = \operatorname{diag}(\alpha_1(\tau), \ldots, \alpha_4(\tau))$$

for every m^j that satisfies the equality $(m^j, \lambda(\tau)) = \lambda_j(\tau)$ for at least one $\tau \in [0, l]$, $|m^j| \geqslant 2$. Taking into account that

$$C^*(\tau)\, d_j(\tau) = \begin{cases} -2b^2(\tau)\, e_j, & j = 1, 2 \\ -2a^2(\tau)\, e_j, & j = 3, 4 \end{cases}$$

where e_j is the j-unit vector we obtain from (1.13) for the vector j-component the unique dependence

$$g_j^{(m^j)}(\tau) = \begin{cases} -2^{-1}\alpha_j^{-1} b^{-2} (h^{(m^j)}, d_j), & j = 1, 2 \\ -2^{-1}\alpha_j^{-1} a^{-2} (h^{(m^j)}, d_j), & j = 3, 4 \end{cases} \tag{1.14}$$

To complete the proof of the theorem it remains to point out that the remaining components of $g^{(m^j)}(\tau)$ are zero.

Thus, selecting in each of systems (ε^k) function $g_k(\tau, u)$ in the form (1.11), we imbed the right-hand sides of these systems in space U. Let us now formulate the solvability conditions for problems (ε^k), $k > 0$ in that space. For this we introduce the following notation. If $h(\tau, u)$ is a polynomial in u, we denote the sum of its terms linear in u by $h^{(1)}(\tau, u)$. Let $h^{(1)}(\tau, u)$ and $g^{(1)}(\tau, u)$ be two linear functions of u. We denote by $\langle h^{(1)}, g^{(1)} \rangle$ the scalar product (for each $\tau \in [0, l]$) in the space $U^{(1)}$ of vector functions linear in u, and define that scalar product as follows:

$$\langle h^{(1)}, g^{(1)}\rangle \equiv \left\langle \sum_{j=1}^{4} h_j(\tau)\, u_j,\ \sum_{j=1}^{4} g_j(\tau)\, u_j \right\rangle = \sum_{j=1}^{4} (h_j(\tau), g_j(\tau))$$

Theorem 2. If all conditions of Theorem 1 are satisfied, then for problem (1.10) to be solvable in space U it is necessary and sufficient that

$$\langle h^{(1)}(\tau, u), d_j(\tau)\, u_j\rangle \equiv 0,\ \forall \tau \in [0, l],\ j = 1, \ldots, 4 \tag{1.15}$$

The proof of this theorem does not differ from that of the similar theorem in /5/.

Conditions (1.2) of imbedding and (1.14) of solvability enable us to obtain single-valued solutions of all iteration problems (ε^k) in class U, and to construct the regularizing system (1.4) of the $(m+1)$ order. We shall prove this on the example of system (ε°):

2. Solvability of the first iteration problem and construction of a first order regularizing system. Solution of problem (ε°) was obtained above in the form of function (1.9), where $\alpha_k(\tau)$ are arbitrary scalar functions that satisfy the initial conditions

$$\alpha_k(0) = \alpha_k^\circ \equiv (C^{-1}(0)\, w^\circ)_k,\ k = 1, \ldots, 4 \tag{2.1}$$

which can be obtained from the equation $w_0(0, 1) = w^\circ$. Let us write down the conditions (1.15) for problem (ε^1) necessary for calculating functions $\alpha_k(\tau)$. Since the sum of linear terms in the right-hand side of system (ε^1) is of the form

$$h^{(1)}(\tau, u) = -\sum_{k=1}^{4} (\alpha_k'(\tau)\, c_k(\tau) + \alpha_k(\tau)\, c_k'(\tau))$$

we write conditions (1.15) in the form of equations

$$(c_k(\tau), d_k(\tau))\, \alpha_k' = -(c_k'(\tau), d_k(\tau))\, \alpha_k,\ k = 1, \ldots, 4 \tag{2.2}$$

Note that $(c_k(\tau), d(\tau)) \neq 0,\ \forall \tau \in [0, l]$, hence using Eqs.(2.2) taking into account the initial conditions (2.1) we uniquely determine functions

$$\alpha_k(\tau) = \overset{\circ}{\alpha}_k \exp\left\{-\int_0^\tau \frac{(c_k'(\tau), d_k(\tau))}{(c_k(\tau), d_k(\tau))}\, d\tau\right\},\quad k = 1, \ldots, 4 \tag{2.3}$$

By the same token we obtain in U the single-valued solution (1.9) of the problem (ε°). The above calculations are, however, valid, if the right-hand side of problem (ε^1) belongs to space U. This can be achieved as indicated above, by a suitable selection of function $g_1(\tau, u)$ in the form of the sum of resonance monomials (1.11). Since the nonlinear part of $F(w_0)$ in (ε^1) has four resonance monomials

$$(2ab\alpha_2\alpha_3 e_2)\, u_2 u_3,\ (2ab\alpha_1\alpha_4 e_2)\, u_1 u_4$$
$$(-b^2\alpha_1^2 e_4)\, u_1^2,\ (-b^2\alpha_2^2 e_4)\, u_2^2$$

then function $g_1(\tau, u)$ is in conformity with (1.14) of the form

$$g_1(\tau, u) = \left(-\frac{a\alpha_2\alpha_3}{\alpha_1 b} e_1\right) u_2 u_3 + \left(-\frac{a\alpha_1\alpha_4}{\alpha_2 b} e_2\right) u_1 u_4 + \frac{b^2\alpha_1^2}{2a^2\alpha_3} e_3 u_1^2 + \frac{b^2\alpha_2^2}{2a^2\alpha_4} e_4 u_2^2 \tag{2.4}$$

and the regularizing first order system assumes the form

$$\begin{aligned}
\varepsilon u_1' &= (-ib)\, u_1 + \varepsilon g_{11}(\tau)\, u_2 u_3,\ u_1(0, \varepsilon) = 1 \\
\varepsilon u_2' &= (+ib)\, u_2 + \varepsilon g_{12}(\tau)\, u_1 u_4,\ \ u_2(0, \varepsilon) = 1 \\
\varepsilon u_3' &= (-ia)\, u_3 + \varepsilon g_{13}(\tau)\, u_1^2,\ \ u_3(0, \varepsilon) = 1 \\
\varepsilon u_4 &= (+ia)\, u_4 + \varepsilon g_{14}(\tau)\, u_2^2,\ \ u_4(0, \varepsilon) = 1
\end{aligned} \tag{2.5}$$
$$\left(g_{11} = -\frac{a\alpha_2\alpha_3}{\alpha_1 b},\ g_{12} = -\frac{a\alpha_1\alpha_4}{\alpha_2 b},\ g_{13} = \frac{b^2\alpha_1^2}{2a^2\alpha_3},\ g_{14} = \frac{b^2\alpha_2^2}{2a^2\alpha_4}\right)$$

System (2.5) is simpler than the singularly perturbed input system (1.1), since it is of the standard differential form. The regularization method of /4/ can be applied to that system, and obtain its asymptotic solution on segment $[0, l]$. However, in this case the regularization method /4/ requires some modification, since it was developed for systems with identical resonances (0.2), while in sysyem (2.5) the monomials $g_{11}u_2u_3$, $g_{12}u_1u_4$, $g_{13}u_1^2$, $g_{14}u_2^2$ correspond to point resonance. The appropriate modification is determined in Sect.3.

3. Asymptotics of standard form. Let us make in (2.5) the substitution of variables

$$u_k = \exp\left\{(-1)^k \frac{i}{\varepsilon}\int_0^\tau (a-b)\,d\tau\right\} v_k, \quad k=1,2 \tag{3.1}$$

$$u_k = \exp\left\{(-1)^k \frac{i}{\varepsilon}\int_0^\tau 2b\,d\tau\right\} v_k, \quad k=3,4$$

which transforms (2.5) into system

$$\varepsilon\frac{dV}{d\tau} = (a-2b)\Lambda_0 V + \varepsilon\exp\left\{\frac{2}{\varepsilon}\int_0^\tau (a-2b)\Lambda_0\,d\tau\right\} g_1(\tau, V),\ V(0,\varepsilon) = \mathbf{1} \tag{3.2}$$

$$V = \{v_1, v_2, v_3, v_4\},\ \Lambda_0 = \operatorname{diag}\{+i, -i, -i, +i\}$$

$$g_1(\tau, V) = \{g_{11}v_2v_3,\ g_{12}v_1v_4,\ g_{13}v_1^2,\ g_{14}v_2^2\}$$

The intention of substitution (3.1) is to pass from the two-frequency system to the single-frequency system (3.2) with the combined frequency $a-2b$. We assume condition (0.3) to be satisfied for $r=1$, and regularize system (3.2), taking into account that the frequency $a-2b$ has a first order zero at point $\tau=\tau_0$. We assume that in (0.3) function $\psi(\tau) > 0$ (although the reasoning would be the same also for $\psi(\tau) < 0$).

We introduce the regularizing function

$$t = \varphi(\tau)/\varepsilon^\alpha,\quad \varphi(\tau_0) = 0 \tag{3.3}$$

For the expansion $V_*(\tau, t, \varepsilon)$ of function $V(\tau, \varepsilon)$ it is reasonable to formulate the following problem

$$\varepsilon\frac{\partial V_*}{\partial\tau} + \varepsilon^{1-\alpha}\frac{\partial V_*}{\partial t}\varphi'(\tau) - (a-2b)\Lambda_0 V_* = \varepsilon\exp\left\{\frac{2}{\varepsilon}\int_0^\tau (a-2b)\Lambda_0\,d\tau\right\} g_1(\tau, V_*),\quad V_*\left(0, \frac{\varphi(0)}{\varepsilon^\alpha}, \varepsilon\right) = \mathbf{1} \tag{3.4}$$

Since we wish to obtain operator $P = \partial/\partial t - t\Lambda_0$ as the principal operator, we separate the two parts of system (3.4) in $\varepsilon^{1-\alpha}\varphi'(\tau)$ and set $(a-2b)/(\varepsilon^{1-\alpha}\varphi') = \varphi/\varepsilon^\alpha = t$. From this, after the separation of variables, we obtain

$$\varepsilon^{1-\alpha} = \varepsilon^\alpha,\quad \varphi\varphi' = a-2b,\quad \varphi(\tau_0) = 0$$

where the first equality shows that $\alpha = 1/2$ and the second that

$$\varphi(\tau) = \left[2\int_{\tau_0}^\tau (a-2d)\,d\tau\right]^{1/2} \tag{3.5}$$

We have, thus, obtained the regularizing function (3.3), where $\varphi(\tau)$ is of the form (3.5) and $\alpha = 1/2$. The expanded system (3.4) now assumes the form

$$\mu\frac{1}{\varphi'}\frac{\partial V_*}{\partial\tau} + PV_* = \mu\frac{1}{\varphi'}\exp\left\{\Lambda_0 t^2 - \Lambda_0\left(\frac{\varphi(0)}{\mu}\right)^2\right\} g_1(\tau, V_*) \tag{3.6}$$

$$V_*(0, \varphi(0)/\mu, \mu) = \mathbf{1},\quad \mu = \sqrt{\varepsilon}$$

Problem (3.6) is regular with respect to μ (as $\mu \to +0$), hence its solution can be sought in the form of series

$$V_*(\tau, t, \mu) = \sum_{k=0}^\infty \mu^k V_k(\tau, t) \tag{3.7}$$

For the coefficients of series (3.7) we obtain the following problems:

$$(\mu^0)\ PV_0 \equiv \partial V_0/\partial t - t\Lambda_0 V_0 = 0,\quad RV_0 \equiv V_0(0, \varphi(0)/\mu) = \mathbf{1}$$

$$(\mu^1)\ PV_1 = -(\varphi')^{-1}\left[\partial V_0/\partial\tau - \exp\left\{\Lambda_0 t^2 - \Lambda_0\left(\frac{\varphi(0)}{\mu}\right)^2\right\} g_1(\tau, V_0)\right]$$

$$RV_1 = 0$$

$$(\mu^2)\ PV_2 = -(\varphi')^{-1}\left[\partial V_1/\partial\tau - \exp\left\{\Lambda_0 t^2 - \Lambda_0\left(\frac{\varphi(0)}{\mu}\right)^2\right\}\frac{\partial g_1(\tau, V_0)}{\partial V} V_1\right]$$

$$RV_2 = 0$$

We shall derive solutions of the iteration problems (μ^k) in space Z of functions of the form

$$V(\tau, t) = \exp\left\{\frac{1}{2}t^2\Lambda_0\right\}\gamma(\tau) + W(\tau, t),\quad \gamma = \{\gamma_1, \ldots, \gamma_4\} \tag{3.8}$$

where $\gamma(\tau)$ is an arbitrary vector function of class $C^\infty[0, l]$ and $W(\tau, t)$ is some function bounded when $t \to +\infty$ and is not a solution of the homogeneous equation $PV = 0$. The conditions of solvability of the iteration systems (μ^k) in class Z is that the right-hand sides of these systems (μ^k) must not contain elements of the kernel of operator P, i.e. of function that is a solution of equation $PV = 0$. System (μ^0) satisfies this condition, and consequently has in class Z a solution of the form

$$V_0(\tau, t) = \exp\left\{\frac{1}{2} t^2 \Lambda_0\right\} \gamma_0(\tau), \qquad \gamma_0 = \{\gamma_{10}, \ldots, \gamma_{40}\} \in C^\infty[0, l] \tag{3.9}$$

where $\gamma_0(\tau)$ is an arbitrary function that satisfied the condition

$$\gamma_0(0) = \exp\left\{-\frac{1}{2}\left[\frac{\varphi(0)}{\mu}\right]^2 \Lambda_0\right\} \mathbf{1} \tag{3.10}$$

To calculate function $\gamma_0(\tau)$ we use the conditions of solvability of problem (μ^1) of class Z. For this we substitute into the right-hand side of system (μ^1) solution (3.9) of problem (μ^0) and collect the coefficients at the exponent $\exp\{{}^1/_2 t^2 \Lambda_0\}$. We obtain

$$(\mu^1)\ PV_1 = -(\varphi')^{-1}\left[\exp\left\{\frac{1}{2} t^2 \Lambda_0\right\} \gamma_0' - \exp\left\{-\left(\frac{\varphi(0)}{\mu}\right)^2 \Lambda_0\right\} g_1(\tau, \gamma_0)\right]$$

For system (μ^1) of class Z to be solvable it is necessary and sufficient that $\gamma_0'(\tau) \equiv 0$, $\forall \tau \in [0, l]$. Taking into account the initial condition (3.10), we obtain from this function $\gamma_0(\tau)$ and, consequently, uniquely determine solution (3.9) of problem (μ^0)

$$V_0(\tau, t) = \exp\left\{\frac{1}{2}\left(t^2 \Lambda_0 - \left[\frac{\varphi(0)}{\mu}\right]^2 \Lambda_0\right)\right\} \mathbf{1} \tag{3.11}$$

Proceeding similarly in the case of problem (μ^1) we obtain for it a solution of the form

$$V_1(\tau, t) = \exp\left\{\frac{1}{2} t^2 \Lambda_0\right\}\left(I(\tau, t) - I\left(0, \frac{\varphi(0)}{\mu}\right)\right) \tag{3.12}$$

$$I(\tau, t) = \left(\int_0^t \exp\left\{-\frac{1}{2}\xi^2 \Lambda_0\right\} d\xi\right) g_1(\tau)/\varphi'(\tau)$$

Restricting it to terms of order $\mu = \sqrt{\varepsilon}$, we obtain for the regularizing system (2.5) the following asymptotic solution:

$$u_{\varepsilon, 1/2}(\tau) = \exp\left\{\frac{1}{\varepsilon}\int_0^\tau \Lambda(\tau) d\tau\right\}\left[\mathbf{1} + \sqrt{\varepsilon}\exp\left\{\frac{1}{2}\left[\frac{\varphi(0)}{\mu}\right]^2 \Lambda_0\right\}\right] \times \left(I\left(\tau, \frac{\varphi(\tau)}{\sqrt{\varepsilon}}\right) - I\left(0, \frac{\varphi(0)}{\sqrt{\varepsilon}}\right)\right)$$

where $\Lambda(\tau)$ denotes $\operatorname{diag}\{-ib, +ib, -ia, +ia\}$. The asymptotic solution of order $\sqrt{\varepsilon}$ of the input problem (1.1) can be obtained by substituting $u = u_{\varepsilon, 1/2}$ into solution (1.9) of problem (ε^0). For instance, for component y we obtain the formula

$$y_{\varepsilon, 1/2}(\tau) = \frac{b(0)}{\sqrt{b(\tau)/b(0)}} \sin\left(\frac{1}{\varepsilon}\int_0^\tau b\, d\tau + \psi_0\right) + \sqrt{\varepsilon}\left(C\cos\left(\frac{1}{\varepsilon}\int_{\tau_0}^\tau b\, d\tau\right) + D\sin\left(\frac{1}{\varepsilon}\int_{\tau_0}^\tau b\, d\tau\right)\right) \tag{3.13}$$

where ψ_0 is some constant, and C and D are the imaginary and real parts of function

$$2i\alpha_1(\tau) b(\tau) \exp\left\{-\frac{i}{\varepsilon}\int_0^{\tau_0} (a - b)\, d\tau\right\}\left(I_1\left(\tau, \frac{\varphi(\tau)}{\sqrt{\varepsilon}}\right) - \right.$$

$$\left. I_1\left(0, \frac{\varphi(0)}{\sqrt{\varepsilon}}\right)\right), \qquad I_1(\tau, t) = \left(\int_0^t \exp\left\{-\frac{i}{2}\xi^2\right\} d\xi\right) \frac{g_{11}(\tau)}{\varphi'(\tau)}$$

For the component x we obtain a formula similar to (3.13). Note that, unlike the composite asymptotics in /1/ which is of different form in each of the three zones indicated above, asymptotics (3.13) is defined by a single formula and is uniformly applicable over the whole length of segment $[0, l]$. The latter means that function $w_0(\tau, u_{\varepsilon, 1/2}(\tau))$ satisfies, under conditions for frequencies a and b defined by Theorem 1, satisfies the inequality

$$\| w(\tau, \varepsilon) - w_0(\tau, u_{\varepsilon, 1/2}(\tau)) \|_{C[0, l]} \leqslant C_0 \varepsilon$$

where $w(\tau, \varepsilon)$ is the exact solution of problem (1.1), and $C_0 > 0$ is a constant independent of ε for fairly small $\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0$.

We would point out that the developed here algorithm is extended to nonlinear systems of the general form.

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